Feel Yourself a Student!

Dear friends, I would like to give to you an interesting and reliable antenna theory. Hours searching in the web gave me lots theoretical information about antennas. Really, at first I did not know what information to choose for ANTENTOP. Finally, I stopped on lectures "Modern Antennas in Wireless Telecommunications" written by Prof. Natalia K. Nikolova from McMaster University, Hamilton, Canada.

You ask me: Why?

Well, I have read many textbooks on Antennas, both, as in Russian as in English. So, I have the possibility to compare different textbook, and I think, that the lectures give knowledge in antenna field in great way. Here first lecture "Introduction into Antenna Study" is here. Next issues of ANTENTOP will contain some other lectures.

So, feel yourself a student! Go to Antenna Studies!

I.G.

My Friends, the above placed Intro was given at ANTENTOP- 01- 2003 to Antennas Lectures.

Now I know, that the Lecture is one of popular topics of ANTENTOP. Every Antenna Lecture was downloaded more than 1000 times!

Now I want to present to you one more very interesting Lecture 17- it is a Lecture Linear Array Theory- Part III. I believe, you cannot find such info anywhere for free! Very interesting and very useful info for every ham, for every radio- engineer.

So, feel yourself a student! Go to Antenna Studies!

I.G.

McMaster University Hall

Prof. Natalia K. Nikolova

Linear Array Theory- Part III

N-element linear array with uniform spacing and non-uniform amplitude: binomial array; Dolph–Tschebyscheff array; directivity and design considerations…

by Prof. Natalia K. Nikolova

LECTURE 17: LINEAR ARRAYS – PART III

(*N-element linear array with uniform spacing and non-uniform amplitude: binomial array; Dolph–Tschebyscheff array; directivity and design considerations.*)

1. Advantages of linear array with non uniform amplitude

The most often met BSAs, classifed according to the type of their excitation amplitude, are:

a) the uniform BSA – relatively high directivity, but the side-lobe levels are high;

b) Dolph–Tschebyscheff (Chebyshev, Чебышев) BSA – for a given number of elements directivity next after that of the uniform BSA, but side-lobe levels are the lowest in comparison with the other two types of arrays for a given directivity.

c) Binomial BSA – does not have good directivity but has very low side-lobe levels (when $d = \lambda/2$, there are no side lobes at all).

2. Array factor (AF) of a linear array with non-uniform amplitude distribution

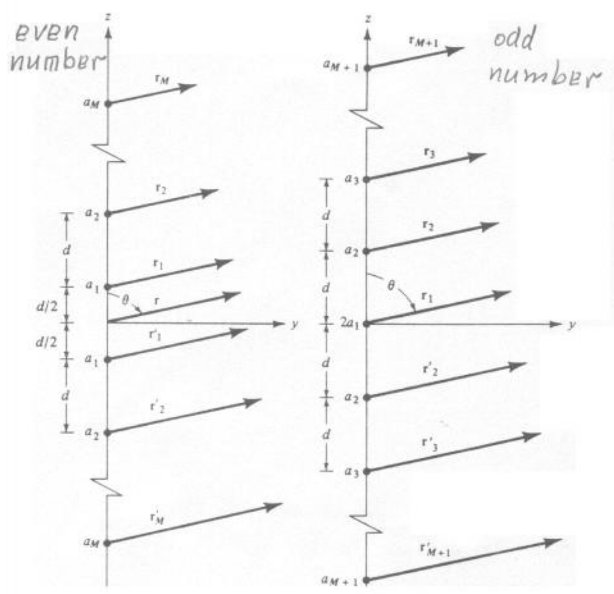
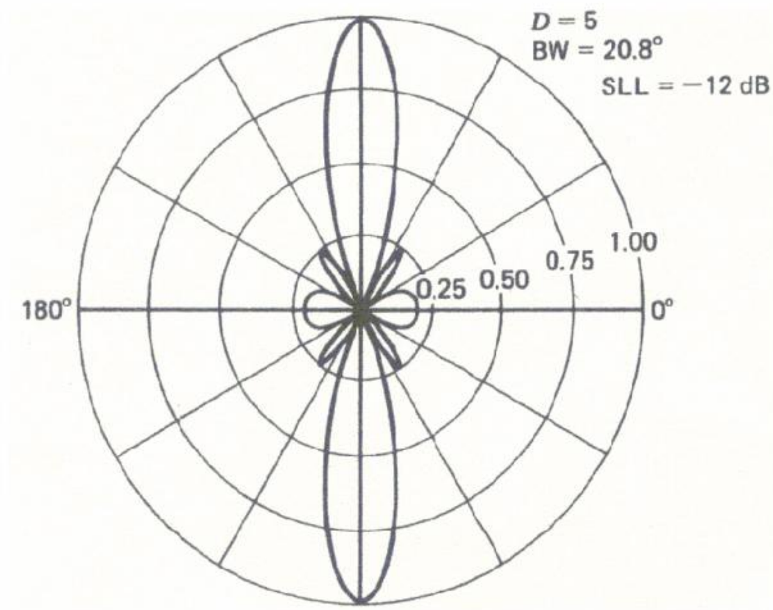


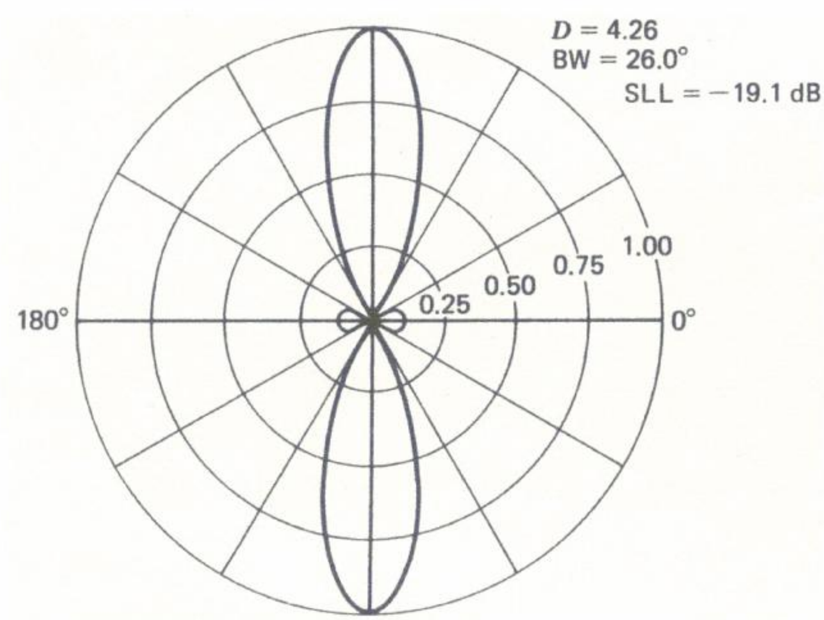
fig. 6.17 pp.291, Balanis

Examples of AFs of arrays of non-uniform amplitude distribution:

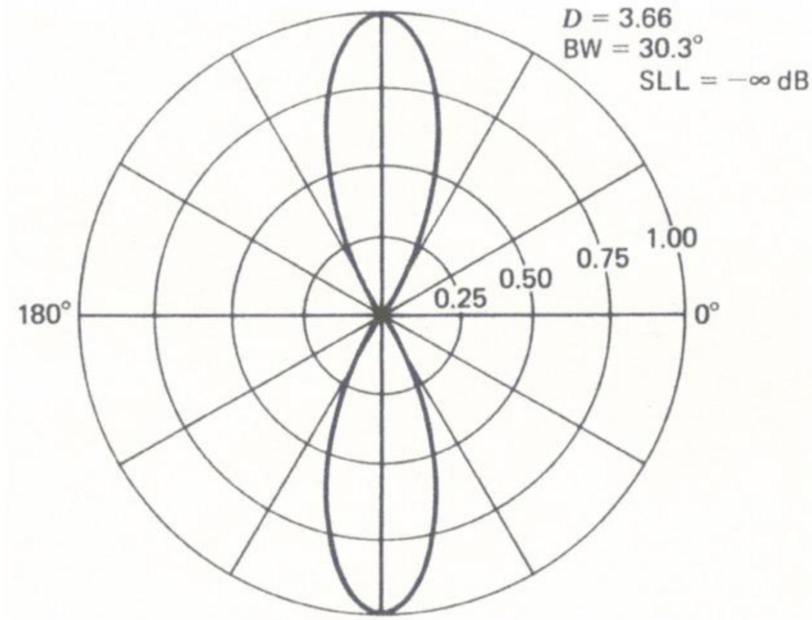
a) uniform amplitude distribution (N=5, $d = \lambda / 2$, $\theta_0 = 90^\circ$)



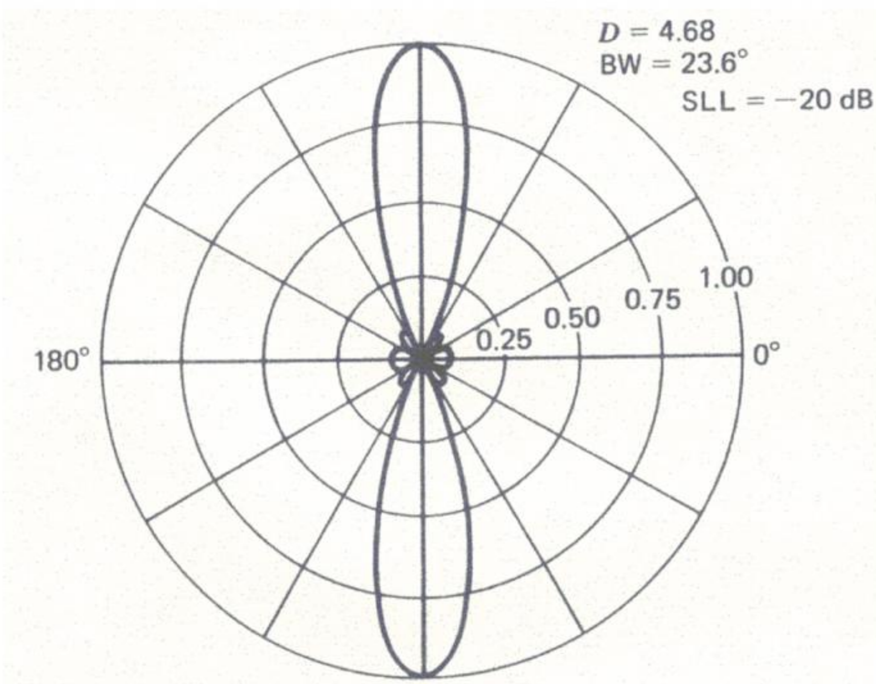
b) triangular (1:2:3:2:1) amplitude distribution (N=5, $d = \lambda / 2$, $\theta_0 = 90^\circ$)



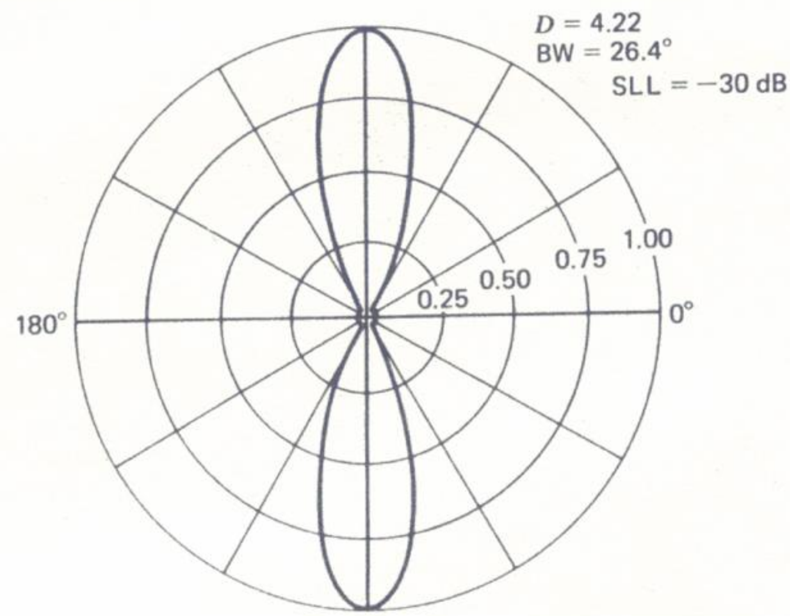
c) binomial (1:4:6:4:1) amplitude distribution (*N*=5, $d = \lambda/2$, $\theta_0 = 90^{\circ}$)



d) Dolph-Tschebyschev (1:1.61:1.94:1.61:1) amplitude distribution (*N*=5, $d = \lambda/2$, $\theta_0 = 90^{\circ}$)



e) Dolph-Tschebyschev (1:2.41:3.14:2.41:1) amplitude distribution (N=5, $d = \lambda/2$, $\theta_0 = 90^\circ$)



Notice that as the current amplitude is tapered more towards the edges of the array, the side-lobes tend to decrease, and the beamwidth tends to increase.

Let us consider a linear array with an even number ($2M$) of elements, located symmetrically along the z-axis, with excitation, which is also symmetrical with respect to $z = 0$. For a broadside array, $(\beta = 0)$:

$$AF^e = a_1 e^{j\frac{1}{2}kd\cos\theta} + a_2 e^{j\frac{3}{2}kd\cos\theta} + \ldots + a_M e^{j\frac{2M-1}{2}kd\cos\theta} + \\ + a_1 e^{-j\frac{1}{2}kd\cos\theta} + a_2 e^{-j\frac{3}{2}kd\cos\theta} + \ldots + a_M e^{-j\frac{2M-1}{2}kd\cos\theta} \qquad (17.1)$$

$$\Rightarrow AF^e = 2\sum_{n=1}^{M} a_n \cos\left[\left(\frac{2n-1}{2}\right)kd\cos\theta\right] \qquad (17.2)$$

If the linear array consists of an odd number (2M+1) of elements, located symmetrically along the z-axis, then the array factor is:

$$AF^o = 2a_1 + a_2 e^{jkd\cos\theta} + a_3 e^{j2kd\cos\theta} + \ldots + a_{M+1} e^{jMkd\cos\theta} + \\ + a_2 e^{-jkd\cos\theta} + a_3 e^{-j2kd\cos\theta} + \ldots + a_{M+1} e^{-jMkd\cos\theta} \quad (17.3)$$

$$\Rightarrow AF^o = 2\sum_{n=1}^{M+1} a_n \cos\left[(n-1)kd\cos\theta\right] \quad (17.4)$$

The factors (2) in equations (17.2) and (17.4) are unimportant for the normalized AF. Equations (17.2) and (17.4) can be re-written as:

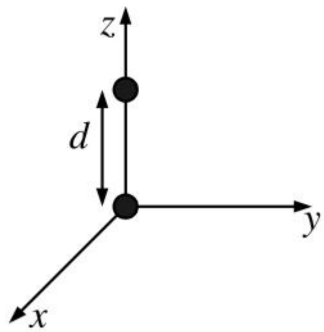
$$AF^e = \sum_{n=1}^{M} a_n \cos\left[(2n-1)u\right], \text{ where } N = 2M \quad (17.5)$$

$$AF^o = \sum_{n=1}^{M+1} a_n \cos\left[2(n-1)u\right], \text{ where } N = 2M+1 \quad (17.6)$$

Here, $u = \dfrac{\pi d}{\lambda}\cos\theta$.

3. Binomial array

The binomial BSA was investigated and proposed by J.S. Stones to synthesize patterns without side lobes. First, consider a 2–element array (along the *z*-axis).

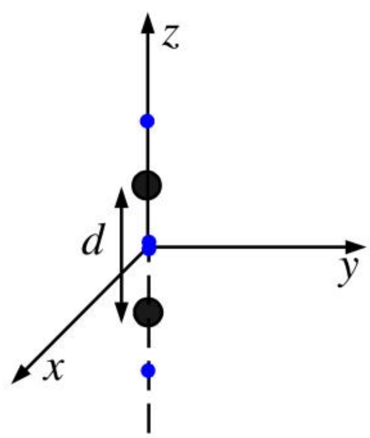


The elements of the array are identical and their excitation is the same. Its array factor is of the form:

$$AF = 1 + Z \text{, where } Z = e^{j\psi} = e^{j(kd\cos\theta + \beta)} \qquad (17.7)$$

If the spacing is $d \le \lambda/2$ and $\beta = 0$ (broad-side maximum), this array will have no side lobes at all.

Second, consider a 2–element array whose elements are identical and the same as the array given above. The distance between the two arrays is again *d*.



This new array has an *AF* of the form:

$$AF = (1+Z)(1+Z) = 1 + 2Z + Z^2 \qquad (17.8)$$

Since $(1+Z)$ has no side lobes, $(1+Z)^2$ will not have side lobes either.

Continuing the process for an *N*-element array produces:

$$AF = (1+Z)^{N-1} \qquad (17.9)$$

If $d \le \lambda/2$, the above *AF* will not have side-lobes regardless of the number of elements *N*. The excitation amplitude distribution can be obtained easily by the expansion of the binome in (17.9). Making use of Pascal's triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

....

the relative excitation amplitudes at each element of an (*N*+1)-element array can be determined. Such an array with a binomial distribution of the excitation amplitudes is called a *binomial array*. The current (excitation) distribution as given by the binomial expansion gives the *relative* values of the amplitudes. It is immediately seen that there is too wide variation of the amplitude, which is the major disadvantage of the BAs. The overall efficiency of such antenna would be very low. Besides, the BA has relatively wide beam. Its HPBW is the largest as compared to this of the uniform BSA or the Dolph–Chebyshev array.

Approximate closed-form expression for the HPBW of a BA with $d = \lambda/2$ is:

$$HPBW = \frac{1.06}{\sqrt{N-1}} = \frac{1.06}{\sqrt{2L/\lambda}} = \frac{1.75}{\sqrt{L/\lambda}} \qquad (17.10)$$

where $L = (N-1)d$ is the array's length. The AFs of 10-element broadside binomial arrays (*N*=10) are given below.

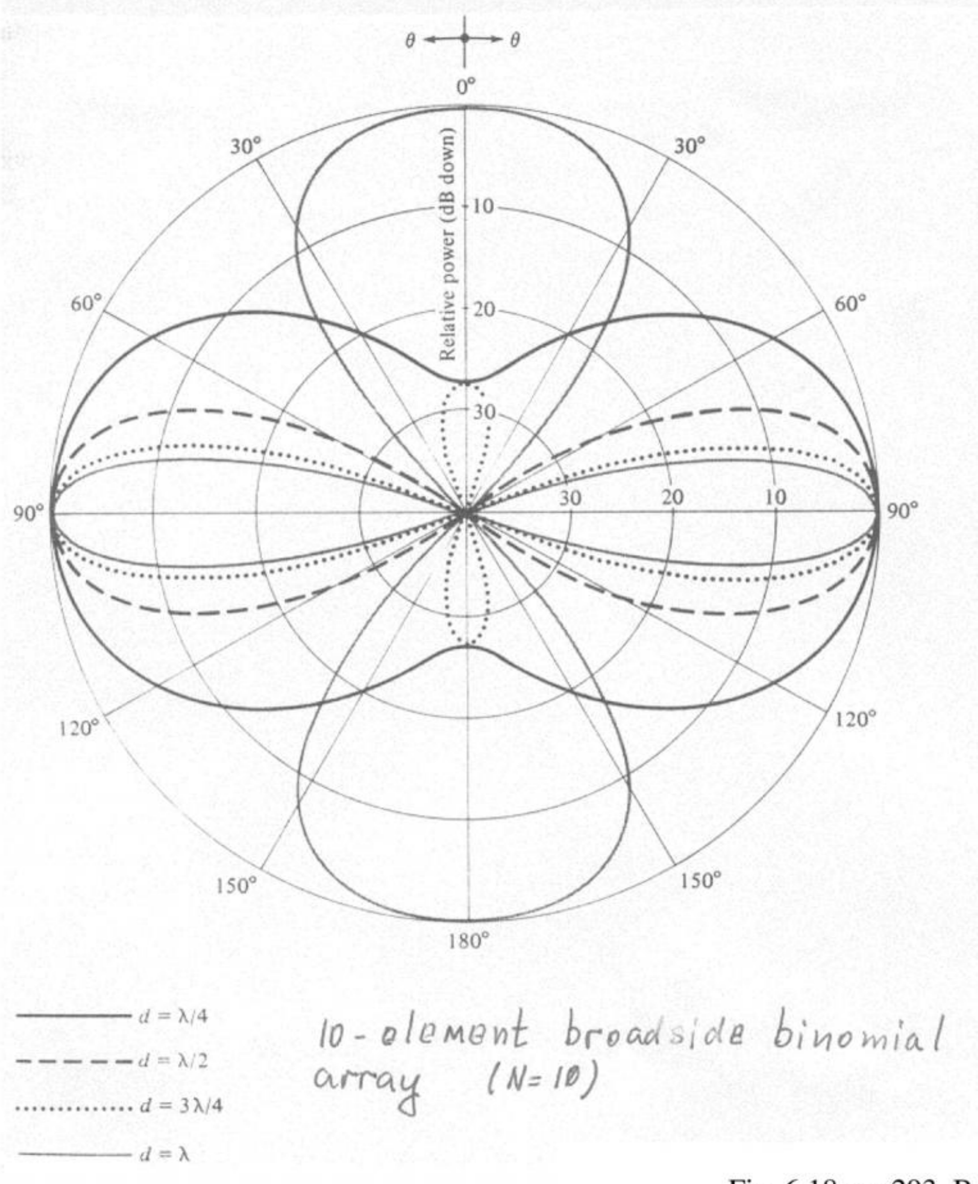


Fig. 6.18, pp.293, Balanis

The directivity of a broadside BA with spacing $d = \lambda/2$ can be calculated from the formula below:

$$D_0 = \frac{2}{\int_0^{\pi} \left[\cos\left(\frac{\pi}{2} \cos\theta \right) \right]^{2(N-1)} d\theta}\Bigg|_{d=\lambda/2} \qquad (17.11)$$

$$D_0 = \frac{(2N-2)(2N-4)...2}{(2N-3)(2N-5)...1} \qquad (17.12)$$

$$D_0 = 1.77\sqrt{N} = 1.77\sqrt{1+2L/\lambda} \qquad (17.13)$$

4. Dolph–Chebyshev (DCA)

Chebyshev≡Tschebyscheff

Dolph proposed (in 1946) a method to design arrays with any desired side-lobe levels and any HPBWs. This method is based on the approximation of the pattern of the array by a Chebyshev polynomial of order *m*, high enough to meet the requirement for the side-lobe levels. Actually, a DCA with no side lobes (side-lobe level of -∞ dB) reduces to the binomial design.

4.1 The Chebyshev polynomials

The Chebyshev polynomials are defined by:

$$T_m(z) = \begin{cases} (-1)^m \cosh(m\cosh^{-1}|z|), & z < -1 \\ \cos(m\cos^{-1} z), & -1 < z < 1 \\ \cosh(m\cosh^{-1} z), & z > 1 \end{cases} \qquad (17.14)$$

A nice feature of Chebyschev polynomials is that $T_m(z)$ of any order m can be derived via a recursion formula, provided $T_{m-1}(z)$ and $T_{m-2}(z)$ are defined.

$$T_m(z) = 2zT_{m-1}(z) - T_{m-2}(z) \tag{17.15}$$

Explicitly, (17.15) produces:

$$\begin{aligned} &m=0, \;\; T_0(z)=1 \\ &m=1, \;\; T_1(z)=z \\ &m=2, \;\; T_2(z)=2z^2-1 \\ &m=3, \;\; T_3(z)=4z^3-3z \\ &m=4, \;\; T_4(z)=8z^4-8z^2+1 \\ &m=5, \;\; T_5(z)=16z^5-20z^3+5z, \;\text{ etc.} \end{aligned} \tag{17.16}$$

If $|z|<1$, then Chebyshev polynomials are related to the cosine functions, see (17.14). One can always expand the function $\cos(mx)$ as a polynomial of $\cos(x)$ of order m, e.g.,

$$\cos 2x = 2\cos^2 x - 1 \tag{17.17}$$

This is done by making use of Euler's formula:

$$(e^{jx})^m = (\cos x + j\sin x)^m = e^{jmx} = \cos(mx) + j\sin(mx) \tag{17.18}$$

Similar relations hold for the hyperbolic cosine function. From the example (17.17), one can see that the Chebyshev argument z is related to the cosine argument x by:

$$z = \cos x \quad \text{or} \quad x = \arccos z \tag{17.19}$$

Then (17.17) can be written as:

$$\cos(2\arccos z) = 2\left[\cos(2\arccos z)\right]^2 - 1$$
$$\Rightarrow \cos(2\arccos z) = 2z^2 - 1 = T_2(z) \qquad (17.20)$$

Compare it with definition (17.14) or with (17.16)-line 3.

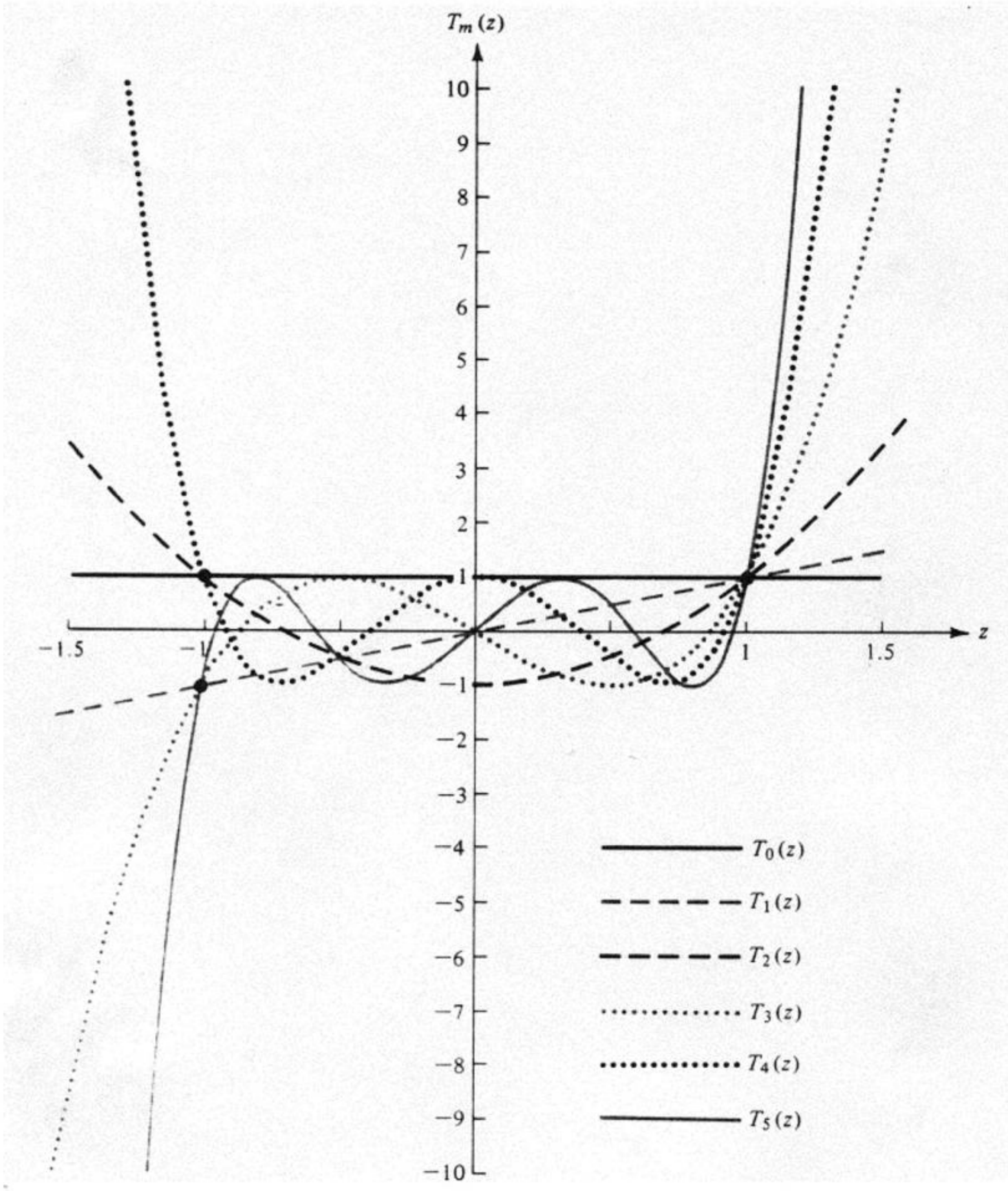


Fig. 6.19, pp.296, Balanis

Properties of Chebyshev polynomials:

1) All polynomials of any order m pass through the point (1,1).
2) Within the range $-1 \le z \le 1$, the polynomials have values within [-1,1].
3) All nulls occur within $-1 \le z \le 1$.
4) The maxima and minima in the $z \in [-1,1]$ range have values $+1$ and -1, respectively.
5) The higher the order of the polynomial, the steeper the slope for $|z| > 1$

4.2 Chebyshev array design

The main goal is to approximate the desired AF with a Chebyshev polynomial such that

- the side-lobe level meets the requirements, and
- the main beam width is small enough.

An array of N elements has an AF, which can be approximated with a Chebyshev polynomial of order m that is always:

$$m = N - 1, \tag{17.21}$$

where: $N = 2M$, if N is even;
$N = 2M + 1$, if N is odd.

In general, for a given side-lobe level, the higher the order of the polynomial, the narrower the beamwidth. But for m>10, the difference is not substantial (see the slopes of $T_m(z)$ in the previous figure).

The AF of an *N*-element array (17.5) or (17.6) will be identical with a Chebyshev polynomial if:

$$T_{N-1}(z) = \begin{cases} \sum_{n=1}^{M} a_n \cos[(2n-1)u], \ \ N = 2M \text{ -even} \\ \sum_{n=1}^{M+1} a_n \cos[2(n-1)u], N = 2M+1\text{-odd} \end{cases} \qquad (17.22)$$

Here, $u = \frac{\pi d}{\lambda}\cos\theta$.

Let the side-lobe level be:

$$R_0 = \frac{E_{\max}}{E_{sl}} = \frac{1}{AF_{sl}} \text{ (voltage ratio)} \qquad (17.23)$$

Then the maximum of T_{N-1} is fixed at an argument z_0, where

$$T_{N-1}^{\max}(z_0) = R_0, \qquad (17.24)$$

where $T_{N-1} > 1$.

Equation (17.24) corresponds to $AF(u) = AF^{\max}(u_0)$. Obviously, z_0 must satisfy the condition:

$$z_0 > 1 \qquad (17.25)$$

Then, the portion of $AF(u)$, which corresponds to $T_{N-1}(z)$ for $|z|<1$, will have levels lower or equal to the specified side-lobe level R_0. This portion of *AF* must correspond to the out-ot-main-beam radiation pattern, i.e. the side lobes. The *AF* is a polynomial of $\cos u$ and the $T_{N-1}(z)$ is a polynomial of z where the limits for z are:

$$-1 \le z \le z_0 \qquad (17.26)$$

Since

$$-1 \le \cos u \le 1 \qquad (17.27)$$

the relation between z and $\cos u$ must be set as:

$$\cos u = \frac{z}{z_0} \qquad (17.28)$$

where $z_0 > 1$.

Array design for an array of *N* elements – general procedure

1) Expand the AF as given by (17.5) or (17.6) by replacing each $\cos(mu)$ term ($m = 1, 2, ..., M$) with the power series of $\cos u$.
2) Determine z_0 such that $T_{N-1}(z_0) = R_0$ (voltage ratio).
3) Substitute $\cos u = z / z_0$ in the AF as found in the previous step.
4) Equate the AF found in Step 3 to $T_{N-1}(z)$ and determine the coefficients for each power of z.

<u>Example:</u> Design a DCA (broadside) of *N*=10 elements with a major-to-minor lobe ratio of $R_0 = 26$ dB. Find the excitation coefficients and form the AF.

<u>Solution:</u>

1. The AF is:

$$AF_{2M} = \sum_{n=1}^{5} a_n \cos[(2n-1)u], \quad u = \frac{\pi d}{\lambda}\cos\theta$$

2. Expand AF_{2M} in terms of $\cos u$:

$$AF_{10} = a_1 \cos u + a_2 \cos 3u + a_3 \cos 5u + a_4 \cos 7u + a_5 \cos 9u$$

Here:

$$\cos 3u = 4\cos^3 u - 3\cos u$$
$$\cos 5u = 16\cos^5 u - 20\cos^3 u + 5\cos u$$
$$\cos 7u = 64\cos^7 u - 112\cos^5 u + 56\cos^3 u - 7\cos u$$
$$\cos 9u = 256\cos^9 u - 576\cos^7 u + 432\cos^5 u - 120\cos^3 u + 9\cos u$$

3. Determine z_0:

$$R_0 = 26 \text{ dB} \Rightarrow R_0 = 10^{26/20} \simeq 20$$
$$\Rightarrow T_9(z_0) = 20$$
$$\cosh\left[9\cosh^{-1}(z_0)\right] = 20$$
$$9\cosh^{-1}(z_0) = \cosh^{-1} 20 = 3.69$$
$$\cosh^{-1}(z_0) = 0.41$$
$$z_0 = \cosh 0.41$$
$$z_0 = 1.08515$$

4. Express the AF in terms of $z = z_0 \cos u$:

$$AF_{10} = \frac{z}{z_0}(a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5) + \frac{z^3}{z_0^3}(4a_2 - 20a_3 + 56a_4 - 120a_5) + \frac{z^5}{z_0^5}(160a_3 - 112a_4 + 432a_5) + \frac{z^7}{z_0^7}(64a_4 - 576a_5) + \frac{z^9}{z_0^9}(256a_5) = \underbrace{9z - 120z^3 + 432z^5 - 576z^7 + 256z^9}_{T_9(z)}$$

5. Finding the coefficients by matching the power terms:

$$256a_5 = 256{z_0}^9 \Rightarrow a_5 = 2.0860$$
$$64a_4 - 576a_5 = -576z_0^7 \Rightarrow a_4 = 2.8308$$
$$16a_3 - 112a_4 + 432a_5 = 432z_0^5 \Rightarrow a_3 = 4.1184$$
$$4a_2 - 20a_3 + 56a_4 - 120a_5 = -120z_0^7 \Rightarrow a_2 = 5.2073$$

$$a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5 = 9z_0^9 \Rightarrow a_1 = 5.8377$$

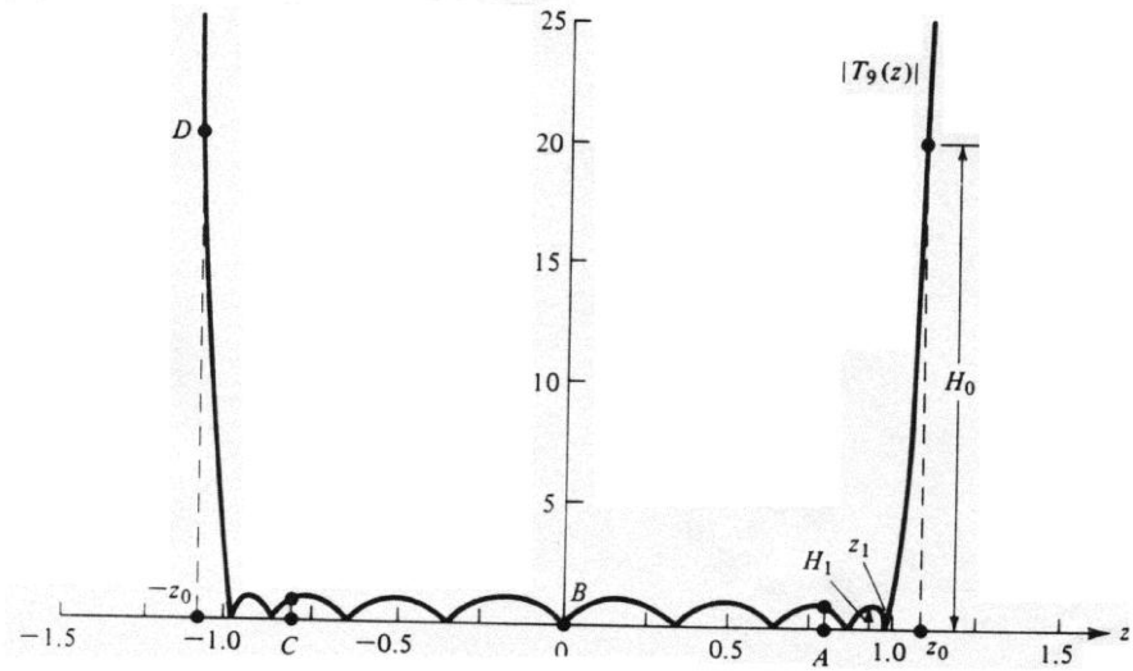


Fig. 6.20b, pp.298, Balanis

6. Normalize coefficients with respect to edge element (N=5):
   $a_5 = 1;\ a_4 = 1.357;\ a_3 = 1.974;\ a_2 = 2.496;\ a_1 = 2.789$

$$AF_{10} = 2.789\cos(u) + 2.496\cos(3u) + 1.974\cos(5u) + 1.357\cos(7u) + \cos(9u)$$

where $u = \dfrac{\pi d}{\lambda}\cos\theta$.

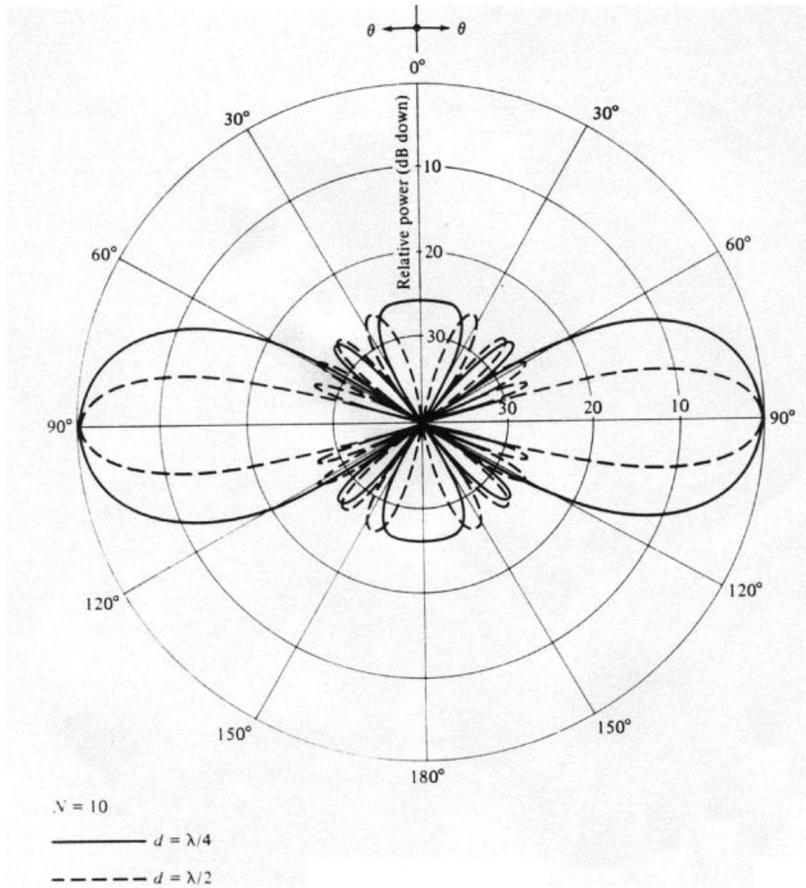


Fig. 6.21, pp.300, Balanis

4.3. The maximum affordable d, $d_{\max}$, for Chebyshev arrays.

This restriction arises from the requirement for a single major lobe – see also equation (17.26).

$$z \geq -1$$

$$\Rightarrow z_0 \cos\left(\frac{\pi d}{\lambda}\cos\theta\right) \geq -1 \qquad (17.29)$$

For a given array, when θ varies from 0° to 180°, the argument z assumes values:

$$\text{from } z = z_0 \cos\frac{\pi d}{\lambda} \text{ to } z = z_0 \cos\left(-\frac{\pi d}{\lambda}\right)$$

The extreme value z to the left of the abscissa is $z = z_0 \cos\frac{\pi d}{\lambda}$ (end-fire directions of the AF, $\theta = 0°$ or $180°$). This value must not go beyond $z = -1$; otherwise minor lobes of levels higher than 1 (higher than R_0) will appear. Therefore, the inequality (17.29) must hold for $\theta = 0°$ or $180°$:

$$z_0 \cos\left(\frac{\pi d_{\max}}{\lambda}\right) \ge -1 \Rightarrow \cos\left(\frac{\pi d_{\max}}{\lambda}\right) \ge -\frac{1}{z_0}$$

Let:

$$\gamma = \arccos\left(\frac{1}{z_0}\right)$$

Then:

$$\frac{\pi d_{\max}}{\lambda} < \pi - \gamma = \pi - \arccos\left(\frac{1}{z_0}\right)$$

$$\Rightarrow \boxed{\frac{d_{max}}{\lambda} < 1 - \frac{1}{\pi}\arccos\left(\frac{1}{z_0}\right)} \qquad (17.30)$$

In the previous example:

$$\frac{d_{max}}{\lambda} < 1 - \frac{1}{\pi}\arccos\left(\frac{1}{1.08515}\right) = 1 - \frac{0.39879}{\pi} = 0.873$$

$$\underline{d_{\max} < 0.873\lambda}$$

5. Directivity of non-uniform arrays

It is difficult to derive closed form expressions for the directivity of non-uniform arrays. Here, we shall derive expressions in the form of series in the most general case of a linear array.

The unnormalized array factor is:

$$AF = \sum_{n=0}^{N-1} a_n e^{j\beta_n} e^{jkz_n \cos\theta} \qquad (17.31)$$

where:

a_n is the amplitude of the excitation of the n-th element;

β_n is the phase angle of the excitation of the n-th element;

z_n is the z-coordinate of the n-th element.

The maximum AF is:

$$AF_{\max} = \sum_{n=1}^{N-1} a_n \tag{17.32}$$

The normalized AF is:

$$AF_n = \frac{AF}{AF_{\max}} = \frac{\sum_{n=0}^{N-1} a_n e^{j\beta_n} e^{jkz_n \cos\theta}}{\sum_{n=1}^{N-1} a_n} \tag{17.33}$$

The beam solid angle is derived as:

$$\Omega_A = 2\pi \int_0^{\pi} \left| AF_n(\theta) \right|^2 \sin\theta d\theta$$

$$\Omega_A = \frac{2\pi}{\left(\sum_{n=1}^{N-1} a_n \right)^2} \sum_{m=0}^{N-1} \sum_{p=0}^{N-1} a_m a_p e^{j(\beta_m - \beta_p)} \int_0^{\pi} e^{jk(z_m - z_p)\cos\theta} \sin\theta d\theta$$

where:

$$\int_0^{\pi} e^{jk(z_m - z_p)\cos\theta} \sin\theta d\theta = \frac{2\sin\left[k\left(z_m - z_p \right) \right]}{k\left(z_m - z_p \right)}$$

$$\Omega_A = \frac{4\pi}{\left(\sum_{n=1}^{N-1} a_n \right)^2} \sum_{m=0}^{N-1} \sum_{p=0}^{N-1} a_m a_p e^{j(\beta_m - \beta_p)} \frac{\sin\left[k\left(z_m - z_p \right) \right]}{k\left(z_m - z_p \right)} \tag{17.34}$$

$$D_0 = \frac{4\pi}{\Omega_A}$$

$$\Rightarrow D_0 = \frac{\left(\sum_{n=1}^{N-1} a_n\right)^2}{\sum_{m=0}^{N-1}\sum_{p=0}^{N-1} a_m a_p e^{j(\beta_m - \beta_p)} \frac{\sin\left[k(z_m - z_p)\right]}{k(z_m - z_p)}} \quad (17.35)$$

For equispaced LA (17.35) reduces to:

$$D_0 = \frac{\left(\sum_{n=1}^{N-1} a_n\right)^2}{\sum_{m=0}^{N-1}\sum_{p=0}^{N-1} a_m a_p e^{j(\beta_m - \beta_p)} \frac{2\sin\left[(m-p)kd\right]}{(m-p)kd}} \quad (17.36)$$

because $z_n = nd$.

For equispaced broadside arrays, where $\beta_m = \beta_p$ for any (m,p), equation (17.36) reduces to:

$$D_0 = \frac{\left(\sum_{n=1}^{N-1} a_n\right)^2}{\sum_{m=0}^{N-1}\sum_{p=0}^{N-1} a_m a_p \frac{\sin\left[(m-p)kd\right]}{(m-p)kd}} \quad (17.37)$$

For equispaced broadside uniform arrays:

$$D_0 = \frac{N^2}{\sum_{m=0}^{N-1}\sum_{p=0}^{N-1} \frac{2\sin\left[(m-p)kd\right]}{(m-p)kd}} \quad (17.38)$$

When the spacing d is a multiple of $\lambda/2$, equation (17.37) reduces to:

$$D_0 = \frac{\left(\sum_{n=1}^{N-1} a_n\right)^2}{\sum_{n=0}^{N-1} (a_n)^2}, \quad d = \frac{\lambda}{2}, \lambda, ... \qquad (17.39)$$

Example: Calculate the directivity of the Dolph–Chebyshev array designed in the previous example if $d = \lambda/2$.

The 10-element DCA has the following amplitude distribution:

$$a_5 = 1; \quad a_4 = 1.357; \quad a_3 = 1.974; \quad a_2 = 2.496; \quad a_1 = 2.798$$

We make use of (17.39):

$$D_0 = \frac{4\left(\sum_{n=1}^{5} a_n\right)^2}{2\sum_{n=0}^{2} (a_n)^2} = 2\frac{(9.625)^2}{20.797} = 8.9 \quad (9.5 \text{ dB}) \qquad (17.40)$$

6. Half-power beamwidth of a BS DCA.

For large DCA with side lobes in the range (-20 to –60) dB, the HPBW can be found by introducing a beam-broadening factor, f, given by:

$$f = 1 + 0.636\left\{\frac{2}{R_0}\cosh\left[\sqrt{(\arccos R_0)^2 - \pi^2}\right]\right\}^2 \qquad (17.41)$$

The HPBW of the DCA is equal to the product of the broadening factor by the HPBW of the respective uniform linear array:

$$HPBW_{DCA} = f \times HPBW_{UA} \qquad (17.42)$$

In (17.41) R_0 denotes the side-lobe level (voltage ratio).